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MATHEMATICS

Simple Exact Solutions to the N. Kowalewski Equations

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For the system of N. Kowalewski equations describing the motion of a heavy rigid body with a fixed point in the case of $B \neq C$, $x_0 \neq 0$, and $y_0 = z_0 = 0$, all 24 families of power-logarithmic expansions (in powers of p) of its solutions were obtained earlier, of which 10 families have $p \to 0$ (tails) and 14 families have $p \to \infty$ (heads). To find finite expansions, we check which of the tail–head pairs give a finite expansion, and which do not. In this way, we obtain all the finite solutions to the N. Kowalewski equations, including all seven previously known solutions and five new ones. All the new solutions are complex. We prove that no other solutions exist that are finite sums of rational powers of p with complex coefficients.

1. In the study of the Euler–Poisson equations describing the motion of a heavy rigid body around a fixed point, success is traditionally associated with the determination of integrable and nonintegrable cases and particular solutions. New possibilities of this study are provided by power geometry [1]. To this day, it has been systematically applied to the system of N. Kowalewski equations

$$f_1 \stackrel{\text{def}}{=} \sigma''\tau + \frac{\sigma'\tau'}{2} + a_1 + a_2\sigma + a_3 p\tau' + a_4\tau + a_5 p^2 = 0,$$
$$f_2 \stackrel{\text{def}}{=} \sigma\tau'' + \frac{\sigma'\tau'}{2} + b_1 + b_2 p\sigma' + b_3\sigma + b_4\tau + b_5 p^2 = 0, \tag{1}$$

where $' \stackrel{\text{def}}{=} \frac{d}{dp}$, p is an independent variable, and $\sigma(p)$ and $\tau(p)$ are dependent variables. According to [2], the Euler–Poisson equations with $B \neq C$, $x_0 \neq 0$, and $y_0 = z_0 = 0$ are reduced to system (1). For N. Kowalewski equations (1), there are two integrable cases (of S. Kovalevskaya and Chaplygin) and nine families of particular solutions (of Steklov [3], Goryachev [4], Chaplygin [5], Kowalewski [2], Appelrott [6], Gorr [7, 8], and Dokshevich and Konosevich-Pozdnyakovich). In nonintegrable cases, all the known particular solutions are finite sums of rational powers of variables of three types: (a) p, (b) p + const, and (c) p^2 + const.

Now there is a possibility of finding all solutions of these kinds. Specifically, 22 families of power-logarithmic (in powers of p) expansions of solutions to the N. Kowalewski system were found in [9–12]. Recently, we have found two more such families and proved that no other solutions exist (see [13]). Altogether, N. Kowalewski equations (1) have 24 families of such expansions of solutions, of which 10 families have $p \to 0$ (tails) and 14 families have $p \to \infty$ (heads). To find finite expansions of solutions of type (a), we verified which of the tail–head pairs are compatible (i.e., give a finite expansion) and which are not. In this way, we obtained all the particular solutions of type (a), including all seven previously known solutions [2–8] and five new ones.

2. The families of expansions of solutions to the N. Kowalewski equations are indexed in the same manner as in [10, 12], except that these families are denoted by $\mathcal{F}$ rather than $\mathcal{H}$. In fact, we use only power expansions with rational power exponents, because expansions with complex or real irrational power exponents and with logarithms cannot be truncated. We consider those complex solutions to the N. Kowalewski equations that are associated with complex solutions to the Euler–Poisson equations. System (1) has two first integrals:

$$f_3 \stackrel{\text{def}}{=} \sigma'\tau - \sigma\tau' + c_1 + c_2 p + c_3 p\sigma + c_4 p\tau + c_5 p^3 = 0,$$
$$f_4 \stackrel{\text{def}}{=} d_1(\sigma')^2\tau + \sigma(\tau')^2 + d_2 + d_3\sigma + d_4\tau + d_5\sigma^2 + d_6 p\sigma'\tau + d_7 p\sigma\tau' + d_8\sigma\tau + d_9\tau^2 + d_{10}p^2 + d_{11}p^2\sigma + d_{12}p^2\tau + d_{13}p^4 = 0. \tag{2}$$

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In (1) and (2), the coefficients a_i, b_i, c_i, and d_i are rational functions of the parameters

$$x \stackrel{\text{def}}{=} \frac{A}{C}, \quad y \stackrel{\text{def}}{=} \frac{B}{C}, \quad z \stackrel{\text{def}}{=} \frac{h}{C}, \quad \lambda \stackrel{\text{def}}{=} \frac{l}{C}, \quad \xi \stackrel{\text{def}}{=} \frac{x_0}{C},$$

where h and l are the values of the energy and moment integrals for the Euler–Poisson equations. Here, x and y are real and satisfy the inequalities

$$x + y \ge 1, \quad x - y \ge -1, \quad y - x \ge -1,$$
$$y \ne 0, \quad y \ne 1,$$

which define a set **D**; z and $\lambda \in \mathbb{C}$; $\xi \in \mathbb{R}$; and $\xi \ne 0$. Systems (1) and (2) have the symmetry

$$(p, \sigma, \tau, x, y, z, \lambda, \xi) \to \left(\bar{p}, -\bar{\tau}, -\bar{\sigma}, \frac{\bar{x}}{\bar{y}}, \frac{1}{\bar{y}}, \frac{\bar{z}}{\bar{y}}, \frac{\bar{\lambda}}{\bar{y}}, \frac{\bar{\xi}}{\bar{y}}\right). \quad (3)$$

Problem. Find all solutions $\sigma(p)$ and $\tau(p)$ to system (1) that are finite sums of rational powers of p:

$$\sigma = \sum_{k=0}^{m} \sigma_k p^{\alpha_k}, \quad \tau = \sum_{l=0}^{n} \tau_l p^{\beta_l}, \quad (4)$$

where α_k and β_l are rational; $\sigma_k, \tau_l \in \mathbb{C}$ are constants; and $\sigma_0, \sigma_m, \tau_0, \tau_n \ne 0$.

By real solutions, we mean those solutions to N. Kowalewski equations (1) that are associated with real solutions to the Euler–Poisson equations; i.e., λ, $z \in \mathbb{R}$ and $(y-1)\sigma$, $(y-1)\tau \ge 0$. A finite solution (4) is treated as known if it has been published somewhere or if it can be obtained from a published solution by symmetry mapping (3) or by taking into account another root of the algebraic equation defining a specific parameter value. A solution that lies on the boundary of the (generating) family of solutions (i.e., lying in its closure) is not treated as independent.

3. To solve the problem, we used all 23 families $\mathcal{F}_1 - \mathcal{F}_{21}$, $\mathcal{F}_{23}$, and $\mathcal{F}_{24}$ of power expansions

$$\sigma = \sigma_0 p^{\alpha} + \sum \sigma_{\alpha+s} p^{\alpha+s},$$
$$\tau = \tau_0 p^{\beta} + \sum \tau_{\beta+s} p^{\beta+s}, \quad s \in \mathbf{K} \quad (5)$$

of solutions to system (1). In (5), the power exponents α, β, and s belong to $\mathbb{R}$; the coefficients σ_0, τ_0, $\sigma_{\alpha+s}$, and $\tau_{\beta+s}$ belong to $\mathbb{C}$ and $\sigma_0, \tau_0 \ne 0$. The families $\mathcal{F}_1$–$\mathcal{F}_{21}$ were found in [9–12]; and $\mathcal{F}_{23}$ and $\mathcal{F}_{24}$, in [13]. It was also proved in [13] that other expansions do not exist. Together with the family $\mathcal{F}_j$, this list also contains its symmetric family $\overline{\overline{\mathcal{F}}}_j$ (according to (3)). Usually, $\overline{\overline{\mathcal{F}}}_j \ne \mathcal{F}_j$, but $\mathcal{F}_3 = \overline{\overline{\mathcal{F}}}_3$, $\mathcal{F}_4 = \overline{\overline{\mathcal{F}}}_4$, and $\mathcal{F}_{19} = \overline{\overline{\mathcal{F}}}_{19}$.

Ten families ($\mathcal{F}_1$–$\mathcal{F}_8$, $\mathcal{F}_{23}$, and $\mathcal{F}_{24}$) have $p \to 0$. They are called tails. Thirteen families ($\mathcal{F}_9$–$\mathcal{F}_{21}$) have $p \to \infty$. They are called heads. Each finite expansion (4) has one tail and one head. Therefore, for each pair of families, i.e., for a tail $\mathcal{F}_i$ and a head $\mathcal{F}_j$, we have to study the intersection

$$\mathcal{F}_i \cap \mathcal{F}_j, \ i \in 1, 2, \ldots, 8, 23, 24; \ j \in 9, 10, \ldots, 21. \quad (6)$$

If it is nonempty, then it gives a finite expansion (4). If it is empty, there is no expansion (4) with the given tail and head. This approach allows us to find all finite expansions (4).

Intersections (6) are analyzed as follows. For each family $\mathcal{F}_m$ of expansions (5), there are known

$$\alpha^{(m)} \stackrel{\text{def}}{=} \alpha, \quad \beta^{(m)} \stackrel{\text{def}}{=} \beta;$$

the set $\mathbf{K}^{(m)} \stackrel{\text{def}}{=} \mathbf{K}$ of values of s, i.e., the sets $\mathbf{K}_\sigma^{(m)}$ and $\mathbf{K}_\tau^{(m)}$ of power exponents $\alpha + s$ and $\beta + s$;

the set $\mathbf{M}^{(m)}$ of admissible values of x, y, z, λ, and ξ;

arbitrary coefficients among $\sigma_{\alpha+s}$ and $\tau_{\beta+s}$.

For each family $\mathcal{F}_m$, any finite number of coefficients $\sigma_{\alpha+s}$ and $\tau_{\beta+s}$ in its expansion (5) can be calculated as rational functions of the parameters. They are sometimes very complicated.

The conditions

$$\alpha^{(i)} \le \alpha^{(j)}, \quad \beta^{(i)} \le \beta^{(j)},$$

$$\mathbf{K}_\sigma^{(i)} \cap \mathbf{K}_\sigma^{(j)} \ne \emptyset, \quad \mathbf{K}_\tau^{(i)} \cap \mathbf{K}_\tau^{(j)} \ne \emptyset, \quad \mathbf{M}^{(i)} \cap \mathbf{M}^{(j)} \ne \emptyset$$

are necessary for the existence of a nonempty intersection (6).

The following step is to check, for each pair $\alpha^{(i)} + s^{(i)} = \alpha^{(j)} + s^{(j)}$, the equality $\sigma_{\alpha^{(i)}+s^{(i)}} = \sigma_{\sigma^{(j)}+s^{(j)}}$; and, similarly, for $\tau_{\beta+s}$.

4. In this way, we have obtained 30 families of finite solutions of form (4). Altogether, there are 16 basic families of solutions, which are denoted by $\mathcal{R}_1$–$\mathcal{R}_{16}$, and another 14 symmetric families (according to (3)), because $\mathcal{R}_7$ and $\mathcal{R}_{13}$ are symmetric to itself: $\mathcal{R}_7 = \overline{\overline{\mathcal{R}}}_7$ and $\mathcal{R}_{13} = \overline{\overline{\mathcal{R}}}_{13}$. The following solutions are new:

$$\mathcal{R}_1\colon x = y = 2, \quad z = 0, \quad \lambda \ne 0,$$

$$\sigma = \frac{\xi\lambda}{8}p^{-1} + \frac{\xi}{2\lambda}p - \frac{p^2}{2}, \quad \tau = -2p^2;$$

$$\mathcal{R}_2\colon y = 1 + \frac{x}{2}, \quad z = \lambda = 0,$$

$$\sigma = -\frac{2\xi^2 p^{-2}}{(x+1)(x-1)^2} + \frac{(1-x)p^2}{2}, \quad \tau = -\frac{p^2}{2};$$

$$\mathcal{R}_3\colon x = y = 2, \quad z = \lambda = 0,$$

$$\sigma = -\frac{2\xi^2 p^{-2}}{3} - \frac{p^2}{2}, \quad \tau = -\frac{p^2}{2};$$

Table 1. Finite expansions with rational power exponents

			$\mathcal{F}_1$	$\mathcal{F}_3$	$\mathcal{F}_4$	$\mathcal{F}_5$	$\mathcal{F}_7$	$\mathcal{F}_{23}$
			$\alpha_1 = 0$	$\alpha_1 = 0$	$\alpha_1 = 2/3$	$\alpha_1 = -1$	$\alpha_1 < 0$	$\alpha_1 \in (1, 2)$
			$\beta_1 = 1$	$\beta_1 = 0$	$\beta_1 = 2/3$	$\beta_1 = 2$	$\beta_1 = 2$	$\beta_1 \in (1, 2)$
$\mathcal{F}_9$	$\alpha_2 > 2$	$\beta_2 = 2$		11 [2]	4 [7]			
				12 [4]				
$\mathcal{F}_{10}$	$\alpha_2 = 2$	$\beta_2 > 2$	8 [2]	$\overline{11}$ $\overline{[2]}$	$\bar{4}$ $\overline{[7]}$		15, 16	
				$\overline{12}$ $\overline{[4]}$				
$\mathcal{F}_{11}$	$\alpha_2 = 2$	$\beta_2 = 2/3$			5			
$\mathcal{F}_{12}$	$\alpha_2 = 2/3$	$\beta_2 = 2$			$\bar{5}$			
$\mathcal{F}_{13}$	$\alpha_2 = 2$	$\beta_2 \in (1, 2)$						
$\mathcal{F}_{14}$	$\alpha_2 \in (1, 2)$	$\beta_2 = 2$						
$\mathcal{F}_{15}$	$\alpha_2 = 2$	$\beta_2 \in (1, 2)$			6 [8]			
$\mathcal{F}_{16}$	$\alpha_2 \in (1, 2)$	$\beta_2 = 2$			$\bar{6}$ $\overline{[8]}$			
$\mathcal{F}_{17}$	$\alpha_2 = 2$	$\beta_2 \in (1, 2)$						
$\mathcal{F}_{18}$	$\alpha_2 \in (1, 2)$	$\beta_2 = 2$						
$\mathcal{F}_{19}$	$\alpha_2 = 2$	$\beta_2 = 2$		13 [3]	7 [5]		2	
$\mathcal{F}_{20}$	$\alpha_2 = 2$	$\beta_2 = 2$	10 [6]	14 [6]		1	3	
$\mathcal{F}_{21}$	$\alpha_2 = 2$	$\beta_2 = 2$	9 [14]	$\overline{14}$ $\overline{[6]}$				

$$\mathcal{R}_5\colon x = y, \quad \lambda = z = 0, \quad \sigma = \sigma_0 p^{2/3} + \sigma_2 p^2, \quad \tau = \tau_0 p^{2/3};$$

$$\tau_0^3 = \frac{81 y \xi^2}{y+2}, \quad \frac{\sigma_0}{\tau_0} = -\frac{y+2}{3y^2}, \quad \sigma_2 = \frac{1-y}{y};$$

$$\mathcal{R}_{15}\colon x = \frac{8}{5}, \quad y = \frac{9}{5}, \quad z = \lambda = 0,$$

$$\sigma = \frac{125}{288}\xi^2 p^{-2} - \frac{1}{18}p^2, \quad \tau = -\frac{1}{2}p^2 - \frac{88}{625\xi^2}p^6;$$

$$\mathcal{R}_{16}\colon x = \frac{14}{9}, \quad y = \frac{16}{9}, \quad z = -\frac{11\tau_1^{-1}}{36}, \quad \lambda = 0,$$

$$\sigma = -\frac{11}{1152}\tau_1^{-2}p^{-2} - \frac{11}{144}\tau_1^{-1} - \frac{p^2}{8}, \quad \tau = -\frac{p^2}{2} - \tau_1 p^4.$$

Here, $\mathcal{R}_3 \subset \mathcal{R}_2$. All these solutions are complex. The solutions $\mathcal{R}_1$ and $\mathcal{R}_3$ belong to the S. Kovalevskaya integrable case; and $\mathcal{R}_5$, to the Chaplygin integrable case for $y = 4$.

In [7] Gorr noted the existence of a complex solution, which can be written in detail as

$$\mathcal{R}_4\colon x = \frac{1268 - 44\sqrt{409}}{375}, \quad y = \frac{241 - 3\sqrt{409}}{250},$$

$$z = 0, \quad \lambda = 0,$$

$$\sigma = \sigma_0 p^{10/3} + \sigma_2 p^2 + \sigma_4 p^{2/3}, \quad \tau = \tau_0 p^2 + \tau_2 p^{2/3},$$

where σ_0 satisfies the equation

$$\sigma_0^3 \xi^2 = \frac{99\,714\,082\,763\,947\,063\sqrt{409} - 2\,016\,592\,523\,367\,734\,611}{1\,305\,600\,000\,000\,000} < 0,$$

and the other coefficients are given by

$$\sigma_2 = \frac{16\,459 - 847\sqrt{409}}{4352},$$

$$\sigma_4 = \frac{1\,211\,697\,549 - 59\,858\,217\sqrt{409}}{189\,399\,040\sigma_0},$$

$$\tau_0 = \frac{7\sqrt{409} - 109}{100}, \quad \tau_2 = \frac{2521\sqrt{409} - 51\,037}{16\,000\sigma_0}.$$

However, the solution $\mathcal{R}_5$ written above was overlooked in [7].

All the remaining known families of solutions have real parts. Moreover, $\mathcal{R}_8$ lies in the closure of $\mathcal{R}_{11}$, and

$\mathcal{R}_9$ and $\mathcal{R}_{10} \subset \mathcal{R}_{14}$. The solutions $\mathcal{R}_9$, $\mathcal{R}_{10}$, and $\mathcal{R}_{14}$ belong to the S. Kovalevskaya integrable case; and $\mathcal{R}_6$, to the Chaplygin integrable case.

For finite expansion (4) written as

$$\sigma = \sigma_0 p^{\alpha_1} + \ldots + \sigma_n p^{\alpha_2}, \quad \tau = \tau_0 p^{\beta_1} + \ldots + \tau_m p^{\beta_2},$$

where $\alpha_1 \le \alpha_2$ and $\beta_1 \le \beta_2$, the results are presented in the table. The columns correspond to the basic tails (without symmetric ones); and the lines, to all the heads. The intersection of the ith column with the jth line gives the index k of the family $\mathcal{R}_k = \mathcal{F}_i \cap \mathcal{F}_j$. For known solutions, the references to publications are given (in square brackets). The blank space in the table means that the corresponding intersection $\mathcal{F}_i \cap \mathcal{F}_j$ is empty. The bar over the index k (i.e., $\bar{k}$) or over a reference indicates the symmetric solution $\overline{\mathcal{R}}_k$ (according to (3)) or the solution symmetric to that given in the referred work.

Here is the main result of this paper.

Theorem. *The system of Eqs.* (1) *has only those exact solutions of type* (4) *that are indicated in the table.*

A detailed presentation of this work can be found in [15].

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